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Discrete-time Volterra chain and classical orthogonal polynomials

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Abstract. A non-isospectral discrete-time Volterra chain (DTVC) is derived from a set of spectral transformations for symmetric orthogonal polynomials (OP). Such DTVC is a natural finite difference analogue of the well known factorization chain for the Schrödinger equation. A class of meromorphic solutions of DTVC is found from an ansatz of semiseparation of variables. The latter yields the very general explicitly known systems of OP—the Askey–Wilson and Askey–Ismail polynomials.

1. Introduction

Monic orthogonal polynomials (OP) of one variable satisfy the three-term recurrence relation [5]

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x) \qquad n = 1, 2, \ldots \tag{1.1}$$

and initial conditions $P_0(x) = 1$, $P_1 = x - b_0$. It is known that OP provide a powerful tool for studying integrable systems with discrete time, see, for example [6, 20]. The key idea consists of the application to OP of spectral transformations going back to Christoffel who discovered, in 1858, the kernel polynomials. The latter are obtained from a given set of OP $P(x)$ by the Christoffel transformation [5, 22]

$$\tilde{P}_n(x; \mu) = \frac{P_{n+1}(x) + C_n P_n(x)}{x - \mu} \tag{1.2}$$

where μ is an arbitrary parameter such that $P_n(\mu) \neq 0$ for all n and $C_n = -P_{n+1}(\mu)/P_n(\mu)$. Iteration of (1.2) leads to a family of orthogonal polynomials $P_n(x; t)$ depending on a discrete time parameter $t = 0, \pm 1, \pm 2, \ldots$ with the shift $t \to t + 1$ defined by

$$P_n(x; t+1) = \frac{P_{n+1}(x; t) + C_n(t+1) P_n(x; t)}{x - \mu(t+1)} \tag{1.3}$$

and $P_n(x; 0) = P_n(x)$, $\mu(1) = \mu$, $C_n(1) = C_n$. The auxiliary spectral parameter $\mu(t)$ in general depends on t. The variables $C_n(t)$ are called superpotentials because there is an evident analogy with supersymmetric quantum mechanics [19]. The inverse transform from the polynomials $P_n(x; t)$ to $P_n(x; t-1)$ is described by the Geronimus transformation [8, 9]

$$P_n(x; t-1) = P_n(x; t) + A_n(t) P_{n-1}(x; t) \tag{1.4}$$

where $A_n(t)$ are new superpotentials.

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Compatibility conditions of (1.3) and (1.4) yield the following relations between superpotentials

$$A_n(t+1)C_{n-1}(t+1) = A_n(t)C_n(t) \tag{1.5}$$

$$C_n(t+1) + A_n(t+1) + \mu(t+1) = A_{n+1}(t) + C_n(t) + \mu(t). \tag{1.6}$$

Recurrence coefficients are expressed in terms of superpotentials in a simple way

$$u_n(t) = A_n(t)C_n(t) \qquad b_n(t) = A_{n+1}(t) + C_n(t) + \mu(t). \tag{1.7}$$

Relations (1.5) and (1.6) coincide with those of the factorization method for the discrete Schrödinger operator [18]; they also describe the non-isospectral discrete-time Toda chain (DTTC) [20]. According the terminology adopted in the theory of integrable systems, the case $\mu(t) =$ constant describes the isospectral discrete-time flow. The Christoffel and Geronimus transformations map OP onto OP, one can abandon such a condition and consider the unrestricted discrete Schrödinger equation instead of (1.1).

The polynomials $P_n(x; t)$ are orthogonal with respect to some measure. For simplicity, we restrict ourselves to the case when the weight function $w(x; t)$ can be defined such that

$$\int_C P_n(x; t)P_m(x; t)w(x; t)\,\mathrm{d}x = h_n(t)\delta_{mn} \tag{1.8}$$

where the normalization constants are $h_0 = 1$, $h_n(t) = u_1(t)u_2(t)\ldots u_n(t)$. In general, both the weight function and the contour of integration C, are complex. However, if the recurrence coefficients $b_n(t)$ are real and $u_n(t)$ are positive, then we have a real positive weight function $w(x; t)$ and the integration is made over an interval of the real axis (Favard's theorem [5]). For the time dependence of the weight functions we have the relation [22]

$$w(x; t+j) = w(x; t)\prod_{k=1}^{j}\frac{(x-\mu(t+k))}{C_0(t+k)} \qquad j > 0. \tag{1.9}$$

All weight functions are normalized by $\int w(x; t)\,\mathrm{d}x = 1$.

Solutions of the factorization chain (or DTTC) (1.5), (1.6) define some systems of OP depending on t in a particular way. As shown in [20], the most general classical OP—the Askey–Wilson polynomials [2]—determine a solution of the DTTC. However, the origin of this elementary function solution for the superpotentials $A_n(t), C_n(t)$ was not clarified. In this work we provide a heuristic derivation of it from the generalized separation of variables in DTVC.

2. Discrete-time Volterra chain and the g-algorithm

Consider a set of symmetric orthogonal polynomials $S_n(x; t)$ satisfying the recurrence relation

$$S_{n+1}(x; t) + v_n(t)S_{n-1}(x; t) = xS_n(x; t) \tag{2.1}$$

with initial conditions $S_0(x) = 1$, $S_1(x) = x$. The Christoffel transformation for symmetric polynomials is, [22]

$$S_n(x; t+1) = \frac{S_{n+2}(x; t) - S_{n+2}(\kappa(t))S_n(x; t)/S_n(\kappa(t))}{x^2 - \kappa^2(t)} \tag{2.2}$$

where $S_n(\kappa(t))$ is a solution of (2.1) for $x = \kappa(t)$—some auxiliary spectral parameter. The recurrence coefficients $v_n(t)$ are transformed as

$$v_n(t+1) = v_n(t)\frac{S_{n-1}(\kappa(t))S_{n+2}(\kappa(t))}{S_n(\kappa(t))S_{n+1}(\kappa(t))}. \tag{2.3}$$

Let us introduce the variables

$$D_n(t) = v_n(t)\frac{S_{n-1}(\kappa(t))}{S_n(\kappa(t))}. \tag{2.4}$$

From the recurrence relation (2.1) we have an important factorization property for the coefficients $v_n(t)$:

$$v_n(t) = D_n(t)(\kappa(t) - D_{n-1}(t)). \tag{2.5}$$

From (2.3) and (2.4) we have also

$$v_n(t+1) = D_n(t)(\kappa(t) - D_{n+1}(t)). \tag{2.6}$$

Comparing (2.5) with (2.6) we arrive at the equation

$$D_n(t+1)(\kappa(t+1) - D_{n-1}(t+1)) = D_n(t)(\kappa(t) - D_{n+1}(t)). \tag{2.7}$$

This equation describes a non-isospectral discrete-time Volterra chain (DTVC). It was derived in [20] from a slightly different approach. The isospectral subcase $\kappa(t) = \text{constant}$ of (2.7) was also discussed in [12]. The possibility that $\kappa(t) \neq \text{constant}$ is very important for the derivation of non-trivial solutions related to the classical OP. Equation (2.7) is very convenient to analyse because it contains only one unknown function of n.

There are two distinct continuous limits of DTVC—in the time t and the lattice n variables. In the first case, one renormalizes $t \to t/h$ and takes the limit $h \to 0$. Then, assuming that $D_n(t+h) \approx D_n(t) + h\dot{D}_n(t)$ and $h\kappa(t) \to -1$, we arrive at the ordinary continuous time Volterra chain

$$\dot{D}_n = D_n(D_{n+1} - D_{n-1}). \tag{2.8}$$

In the second case, one takes $z = nh$ fixed, $h \to 0$. Then, assuming that in this limit $D_n = 1 + hf(z;t) + \mathrm{O}(h^3)$, where $f(z;t)$ are the continuous coordinate superpotentials, and $D_{n\pm1} \approx 1 + hf(z;t) \pm h^2 f_z(z;t)$, $\kappa(t) \approx 2 - h^2\nu(t)$, we arrive at the equation

$$f_z(z;t) + f_z(z;t+1) + f^2(z;t) - f^2(z;t+1) = \nu(t+1) - \nu(t) \tag{2.9}$$

which is nothing else than the factorization chain for the Schrödinger equation [13, 17]

$$-\psi_{zz}(z;t) + (f^2(z;t) - f_z(z;t) + \nu(t))\psi(z;t) = \lambda\psi(z;t).$$

Thus the DTVC (2.7) can be considered as a natural discretization of two well-known differential-difference chains.

DTVC (2.7) and system (1.5), (1.6) are deeply related to each other. From a given solution $D_n(t), \kappa(t)$ of the DTVC one can construct solutions $A_n(t), C_n(t), \mu(t)$ of DTTC via the following mapping [20]:

$$\begin{aligned} A_n(t) &= -D_{2n-1}(t)D_{2n}(t) \\ C_n(t) &= -(\kappa(t) - D_{2n}(t))(\kappa(t) - D_{2n+1}(t)) \\ \mu(t) &= \text{constant} + \kappa^2(t). \end{aligned} \tag{2.10}$$

These relations can be derived from the correspondence between the symmetric and non-symmetric OP [5]. Indeed, the polynomials

$$P_n(y;t) = S_{2n}(x;t+1) \qquad y = x^2 \tag{2.11}$$

satisfy the recurrence relation (1.1) with the coefficients

$$u_n(t) = v_{2n-1}(t+1)v_{2n}(t+1) \qquad b_n(t) = v_{2n}(t+1) + v_{2n+1}(t+1). \tag{2.12}$$

Formulae (2.10) are compatible with (2.12) (the mapping (2.12) is known to relate the ordinary Volterra chain with the Toda chain [23]).

In a similar way, the polynomials $P_n(x^2;t) = x^{-1}S_{2n+1}(x;t+1)$ satisfy (1.1) with $u_n(t) = v_{2n}(t+1)v_{2n+1}(t+1)$, $b_n(t) = v_{2n+1}(t+1) + v_{2n+2}(t+1)$. This corresponds to the following relation between the soultions of DTVC and DTTC:

$$A_n(t) = -D_{2n}(t)D_{2n+1}(t) \qquad C_n(t) = -(\kappa(t) - D_{2n+1}(t))(\kappa(t) - D_{2n+2}(t)) \tag{2.13}$$

and $\mu(t) = \text{constant} + \kappa^2(t)$.

The g-algorithm was proposed by Bauer [3] in the theory of rational approximation. In the core of this algorithm lie the so-called *g-rhombus rules* which, in the Bauer's original form, look as follows

$$g_{2n-2}(t+1)(c(t+1) - g_{2n-1}(t+1)) = g_{2n}(t)(c(t) - g_{2n-1}(t)) \tag{2.14}$$

$$g_{2n-1}(t+1)(1 - g_{2n}(t+1)) = g_{2n+1}(t)(1 - g_{2n}(t)). \tag{2.15}$$

The coefficients $g_n(t)$ are well known in the theory of continued fractions as g-decompositions of the Stieltjes S-fraction, see, e.g. [24]. The numbers $c(t)$ determine positions of poles of the corresponding continued fraction [3]. If one changes the variables $c(t) = \kappa^2(t)$ and

$$g_{2n}(t) = \frac{\kappa(t) - D_{2n}(t)}{\kappa(t)} \qquad g_{2n+1}(t) = \kappa(t)(\kappa(t) - D_{2n+1}(t)) \tag{2.16}$$

then for $\kappa(t) \neq 0$ the Bauer's g-rhombus rules (2.14), (2.15) become equivalent to DTVC (2.7).

The correspondence (2.13) is contained in the Bauer's work as well: his formulae (37), (38) coincide with (2.13) recast in our notations. However, the fact that two g-algorithm equations can be unified into one is essential (especially for the considerations given below) and it was not established in [3]. The DTTC equations (1.5), (1.6) are referred to in [3] as the 'modified qd-algorithm'. The ordinary qd-algorithm in the form proposed by Rutishauser [16] is equivalent to the isospectral case of DTTC (cf [12]). As far as we know, the non-isospectral case was not further exploited in the literature neither in the study of integrable systems, nor in the applications to numerical algorithms.

3. Semiseparation of variables

We would like to construct a class of solutions $D_n(t)$ of the DTVC, which appear from a generalized separation (semiseparation) of variables. One can arrive at the corresponding ansatz from the analysis of meromorphic solutions admitting simple poles and zeros.

Let $D_n(t)$ be a meromorphic function of both variables n and t (i.e. discrete potentials $u_n(t)$ and $b_n(t)$ are meromorphic too). For simplicity, we demand that $D_n(t)$ has only simple poles in the complex n-plane and no pole in (2.7) is cancelled by a zero. Substituting the expansion

$$D_n(t) = \sum_{k=1}^{\infty} \frac{\gamma_k(t)}{n - \alpha_k(t)} + \text{an entire function}$$

into the DTVC, one obtains the condition of cancellation of all poles

$$\prod_{k,l=1}^{\infty} (n - \alpha_k(t+1))(n - 1 - \alpha_l(t+1)) = \prod_{k,l=1}^{\infty} (n - \alpha_k(t))(n + 1 - \alpha_l(t)).$$

One easily sees that these constraints are resolved (we do not discuss the uniqueness) when the poles of $D_n(t)$ depend only on the combination $n + t$. That is, we can set $D_n(t) \propto 1/g(n+t)$, where $g(x)$ is an entire function.

Substitute now into DTVC the ansatz $D_n(t) = r(n,t)\prod_{k=1}^{\infty}(n-\beta_k(t))$, where $r(n,t)$ is a meromorphic function without zeros for finite n. Then either the zeros of $D_n(t)$ are cancelled by the zeros of $D_n(t+1)$, or one has a much more complicated condition involving the unknown function $\kappa(t)$. We shall restrict ourselves to the first case, which means that the position of zeros is determined by an entire function a_n which does not depend on t, i.e. we set $D_n(t) \propto a_n/g(n+t)$. Observe the striking resemblance between this requirement and what is usually a condition for separation of variables. Demanding that the missing factor depends only on t, we come to the ansatz of generalized separation (semiseparation) of variables

$$D_n(t) = \frac{a_n\sigma(t)}{g(n+t)}. \tag{3.1}$$

However, this is not as general as we want it. Due to the odd–even index split of variables (2.13), it is natural to consider $D_{2n}(t)$ and $D_{2n+1}(t)$ as two different functions meromorphic in both variables t and n. The structure of DTVC shows that such an assumption does not effect the condition of cancellation of zeros that we have taken. But it might touch the dynamics of poles. Therefore, we apply the ansatz (3.1) to both $D_{2n}(t)$ and $D_{2n+1}(t)$ independently, but assume that the poles are fixed by a single function $g(x)$. This means that we take

$$D_{2n}(t) = \frac{\sigma_1(t)c_n}{g(2n+t)} \qquad D_{2n+1}(t) = \frac{\sigma_2(t)d_n}{g(2n+t+1)} \tag{3.2}$$

where the functions $\sigma_i(t)$, $i = 1, 2$ depend only on t and c_n, d_n depend only on n. Of course, the considerations given above should be viewed only as a heuristic way of constructing a particular class of meromorphic solutions of (2.7) via (3.2). The following theorem was announced in [20], in the present paper we give its detailed proof.

Theorem. The ansatz of semiseparation of variables (3.2) for the DTVC is resolved completely in terms of elementary functions. It yields recurrence coefficients of two distinguished classes of orthogonal polynomials. The first one corresponds to the (associated) Askey–Wilson polynomials [2, 10, 14], and the second one corresponds to the (associated) Askey–Ismail polynomials [1].

Let us substitute expressions (3.2) into (2.7). This gives the following equations

$$\sigma_1(t+1)(\kappa(t+1)g(2n+t) - \sigma_2(t+1)d_{n-1}) = \sigma_1(t)(\kappa(t)g(2n+t+1) - \sigma_2(t)d_n) \tag{3.3}$$

$$\sigma_2(t+1)(\kappa(t+1)g(2n+t+1) - \sigma_1(t+1)c_n) = \sigma_2(t)(\kappa(t)g(2n+t+2) - \sigma_1(t)c_{n+1}). \tag{3.4}$$

It is not difficult to integrate each of them once

$$\xi(t)g(2n+t+1) - \zeta(t)d_n = E(t+n) \tag{3.5}$$

$$\eta(t)g(2n+t+2) - \zeta(t)c_{n+1} = F(t+n) \tag{3.6}$$

where $E(x)$ and $F(x)$ are some functions (first integrals) and

$$\xi(t) = \sigma_1(t)\kappa(t) \qquad \eta(t) = \sigma_2(t)\kappa(t) \qquad \zeta(t) = \sigma_1(t)\sigma_2(t). \tag{3.7}$$

Surprisingly, all the unknowns $\xi(t), \eta(t), \zeta(t),\ c_n, d_n, F(x), E(x)$ entering (3.5), (3.6) are determined uniquely.

Before starting the analysis, let us write the expressions for superpotentials $A_n(t)$, $C_n(t)$ using the formulae (2.13):

$$A_n(t) = -\frac{\zeta(t)c_n d_n}{g(2n+t)g(2n+t+1)} \qquad C_n(t) = -\frac{E(t+n)F(t+n)}{\zeta(t)g(2n+t+1)g(2n+t+2)}. \tag{3.8}$$

Recurrence coefficients for $S_n(x;t)$ are fixed as

$$v_{2n}(t) = \frac{c_n E(t+n-1)}{g(2n+t)g(2n+t-1)} \qquad v_{2n+1}(t) = \frac{d_n F(t+n-1)}{g(2n+t)g(2n+t+1)}. \tag{3.9}$$

These formulae will be useful for relating our solutions to some orthogonal polynomials.

The structure of (3.5) and (3.6) coincide, i.e. it is sufficient to consider only one of them, say (3.5). Taking the initial equation (3.3)

$$\xi(t)g(2n+t+1) - \zeta(t)d_n = \xi(t+1)g(2n+t) - \zeta(t+1)d_{n-1} \tag{3.10}$$

and excluding d_{n-1}, we arrive at the equation

$$\begin{aligned} r(t)d_n = {} & \xi(t)\zeta(t+1)g(2n+t-1) + \xi(t)\zeta(t)g(2n+t+1) \\ & -(\zeta(t)\xi(t+1) + \xi(t-1)\zeta(t+1))g(2n+t) \end{aligned} \tag{3.11}$$

where $r(t) = \zeta^2(t) - \zeta(t+1)\zeta(t-1)$.

Further analysis depends on whether $\zeta(t) = \zeta_0 e^{\omega t}$ or not. It will be shown that the $\zeta(t) = \zeta_0 e^{\omega t}$ case leads to the general Askey–Wilson polynomials [2]. If $\zeta(t)$ is not a pure exponential function of t, then we arrive at the Askey–Ismail polynomials [1], which may also be called symmetric q-Pollaczek polynomials [4].

4. Investigation of the Askey–Wilson case

For $\zeta(t) = \zeta_0 e^{\omega t}$ (we shall normalize $\zeta_0 = 1$ by redefining d_n, say), solution of (3.11) is quite easy. Since the factor in front of d_n on the left-hand side is equal to zero, the complete separation of variables is reached on the right-hand side:

$$\frac{g(x+1) + e^{\omega} g(x-1)}{g(x)} = \frac{\xi(t+1) + e^{\omega}\xi(t-1)}{\xi(t)} = \text{constant}. \tag{4.1}$$

A general solution of these equations is obviously given by

$$\xi(t) = \xi_1 e^{\omega_1 t} + \xi_2 e^{\omega_2 t} \qquad g(x) = g_1 e^{\omega_1 x} + g_2 e^{\omega_2 x} \tag{4.2}$$

where ξ_i, g_i are arbitrary constants and $\omega_1 + \omega_2 = \omega$. Substituting (4.2) into (3.5) and comparing the terms depending on t (for fixed n) we find the general form of $E(t+n)$:

$$E(y) = \epsilon_1 e^{2\omega_1 y} + \epsilon_2 e^{2\omega_2 y} + \epsilon_3 e^{\omega y} \tag{4.3}$$

where $y = n + t$, and ϵ_i, $i = 1, 2, 3$ are some constants. For d_n we obtain from (3.5) an expression of the similar structure:

$$d_n = \delta_1 e^{2\omega_1 n} + \delta_2 e^{2\omega_2 n} + \delta_3 e^{\omega n}. \tag{4.4}$$

Similarly, from (3.6) we find $c_n, \eta(t)$ and $F(y)$:

$$\begin{aligned} & c_n = \gamma_1 e^{2\omega_1 n} + \gamma_2 e^{2\omega_2 n} + \gamma_3 e^{\omega n} \qquad \eta(t) = \eta_1 e^{\omega_1 t} + \eta_2 e^{\omega_2 t} \\ & F(y) = \phi_1 e^{2\omega_1 y} + \phi_2 e^{2\omega_2 y} + \phi_3 e^{\omega y}. \end{aligned} \tag{4.5}$$

From (3.5) and (3.6) we find relations between the parameters

$$\begin{aligned} &\epsilon_1 = \xi_1 g_1 e^{\omega_1} \qquad \epsilon_2 = \xi_2 g_2 e^{\omega_2} \qquad \epsilon_3 = -\delta_3 \\ &\delta_1 = \xi_2 g_1 e^{\omega_1} \qquad \delta_2 = \xi_1 g_2 e^{\omega_2} \\ &\phi_1 = \eta_1 g_1 e^{2\omega_1} \qquad \phi_2 = \eta_2 g_2 e^{2\omega_2} \qquad \phi_3 = -\gamma_3 e^{\omega} \\ &\gamma_1 = \eta_2 g_1 \qquad \gamma_2 = \eta_1 g_2. \end{aligned} \tag{4.6}$$

The initial condition $A_0(t) = 0$ usually simplifies the structure of OP, but in general it is not obligatory (e.g. for the associated polynomials). We impose this condition in order to fix one of the parameters—it is equivalent to the constraint $c_0 = 0$, or $d_0 = 0$. Only the first possibility is relevant, i.e. we have $\gamma_1+\gamma_2+\gamma_3 = 0$ (the $d_0 = 0$ case leads to $v_1 = 0$ in the recurrence relation (2.1) which is not allowed for OP). From the derived constraints, we can express all our parameters in terms of nine constants: $\gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, g_1, g_2, \omega_1, \omega_2$. As a result, the superpotentials $A_n(t)$ and $C_n(t)$ take the form

$$A_n(t) = -\frac{\gamma_1\delta_1 q^t (1-q^n)(1-\frac{\gamma_2}{\gamma_1}q^n)(1+\frac{\delta_2}{\delta_1}q^{2n}+\frac{\delta_3}{\delta_1}q^n)}{e^{\omega_1}(1+\frac{g_2}{g_1}q^{2n+t})(1+\frac{g_2}{g_1}q^{2n+t+1})} \tag{4.7}$$

$$C_n(t) = -\frac{\gamma_2\delta_2(1+\frac{g_2}{g_1}q^{n+t+1})(1+\frac{\gamma_1 g_2}{\gamma_2 g_1}q^{n+t+1})(1-\frac{\delta_3 g_2}{\delta_2 g_1}q^{n+t+1}+\frac{\delta_1 g_2^2}{\delta_2 g_1^2}q^{2(n+t+1)})}{e^{\omega_2}q^t(1+\frac{g_2}{g_1}q^{2n+t+1})(1+\frac{g_2}{g_1}q^{2n+t+2})} \tag{4.8}$$

where $q = \exp(\omega_2 - \omega_1)$. We also need the auxiliary spectral parameter $\kappa(t)$. From (3.7) we obtain

$$\kappa^2(t) = \frac{\xi(t)\eta(t)}{\zeta(t)} = \frac{\gamma_2\delta_2}{e^{\omega_2}g_2^2}q^{-t} + \frac{\gamma_1\delta_1}{e^{\omega_1}g_1^2}q^t + \frac{\delta_1\gamma_2 e^{-\omega_1}+\delta_2\gamma_1 e^{-\omega_2}}{g_1 g_2}. \tag{4.9}$$

Let us look at expressions (4.7), (4.8). Despite the presence of nine parameters, there are only five independent constants in our formulae, namely, $\frac{g_2}{g_1}$, $\frac{\gamma_1}{\gamma_2}$, $\frac{\delta_3}{\delta_2}$, $\frac{\delta_1}{\delta_2}$ and q. They determine zeros of the numerators and denominators. The other four free parameters enter as normalization factors in $A_n(t)$ and $C_n(t)$ and, hence, may be arbitrary.

Now we may compare (4.7), (4.8) with the superpotentials of the general Askey–Wilson polynomials derived in [20]:

$$A_n(t) = -\frac{aq^t(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n+t-2})(1-abcdq^{2n+t-1})} \tag{4.10}$$

$$C_n(t) = -\frac{(1-abcdq^{n+t-1})(1-abq^{n+t})(1-acq^{n+t})(1-adq^{n+t})}{aq^t(1-abcdq^{2n+t-1})(1-abcdq^{2n+t})} \tag{4.11}$$

where a, b, c, d, q are parameters of the polynomials. We see that the Askey–Wilson parameters a, b are related to ours as follows

$$a = \frac{g_2}{g_1}\sqrt{\frac{q\gamma_1\delta_1}{\gamma_2\delta_2}} \qquad b = -\sqrt{\frac{q\gamma_1\delta_2}{\gamma_2\delta_1}}. \tag{4.12}$$

The parameters c and d are found in terms of our constants from the relations

$$c + d = \delta_3\sqrt{\frac{q\gamma_2}{\gamma_1\delta_1\delta_2}} \qquad cd = \frac{q\gamma_2}{\gamma_1}. \tag{4.13}$$

Formally, one also has to impose the condition $\gamma_1\gamma_2\delta_1\delta_2 = e^{\omega_1+\omega_2}g_1^2 g_2^2$, which is not essential because it can be satisfied by the appropriate homogeneous rescaling of superpotentials.

The explicit form of the Askey–Wilson polynomials is [2, 7]

$$P_n(x) = \frac{(ab;q)_n(ac;q)_n(ad;q)_n}{a^n(abcdq^{n-1};q)_n}\, {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right) \tag{4.14}$$

where $x = z + z^{-1}$. We use the standard notations [7] for the basic hypergeometric series

$${}_r\varphi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \dots (a_r;q)_n}{(q;q)_n(b_1;q)_n \dots (b_s;q)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n$$

and the q-shifted factorial, $(a;q)_0 = 1$, $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$, $n > 0$. The dependence on time t may be restored by replacing a by aq^{t+1}.

Thus we have related the first set of DTVC solutions to the general Askey–Wilson potentials with the arbitrary basic parameter q. The associated polynomials [10, 14] (or general functional solutions of the corresponding difference equation) enter the scheme as well, since they correspond to a homogeneous shift of n by a constant in the solution of DTVC. There are also finite-dimensional systems, appearing from a quantization of parameters. In particular, such a situation takes place when q is a primitive root of unity, leading to some non-trivial trigonometric identities [21].

Let us stress that our formulae provide a symmetric [5] (or supersymmetric [19]) representation of the recurrence relation for the Askey–Wilson polynomials. To find the latter is a non-trivial task. The symmetric polynomials $S_n(x;t)$ with recurrence coefficients $v_n(t)$, defined by the substitution of our expressions for $D_n(t)$ into (2.5), contain all relevant information. In this case, the difference operator on the left-hand side of (2.1) is a square root of the Askey–Wilson difference operator, which is clear from (2.12). This symmetric representation was used in [26] for generating new polynomials orthogonal on the unit circle or arcs.

Let us now discuss the isospectral subcase of the derived solution of DTVC. From (4.9), we see that the auxiliary spectral parameter $\kappa(t) = \text{constant} \neq 0$ (i.e. $\mu(t) = \text{constant}$) if $\gamma_2 = \delta_1 = 0$. In this case the discrete time evolution evidently does not change the spectrum of the model.

From (4.7) and (4.8) we have the following expressions for the superpotentials

$$A_n(t) = -\frac{\delta_3\gamma_1 q^{t+n}(1-q^n)(1+\frac{\delta_2}{\delta_3}q^n)}{\mathrm{e}^{\omega_1} g_1^2(1+\frac{g_2}{g_1}q^{2n+t})(1+\frac{g_2}{g_1}q^{2n+t+1})} \tag{4.15}$$

$$C_n(t) = -\frac{\delta_2\gamma_1 q^n(1+\frac{g_2}{g_1}q^{n+t+1})(1-\frac{\delta_3 g_2}{\delta_2 g_1}q^{n+t+1})}{\mathrm{e}^{\omega_1} g_1 g_2(1+\frac{g_2}{g_1}q^{2n+t+1})(1+\frac{g_2}{g_1}q^{2n+t+2})}. \tag{4.16}$$

These functions define the little q-Jacobi polynomials with the parameters $a = \delta_3 g_2/\delta_2 g_1$, $b = -\delta_2/\delta_3$:

$$P_n(x;t) = \frac{(-1)^n q^{n(n-1)/2}(aq^{t+2};q)_n}{(abq^{n+t+2};q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+t+2} \\ aq^{t+2} \end{matrix}; q, qx\right).$$

This can be seen by a direct comparison of the derived superpotentials with those given in [15]. Formally one also has to impose the constraint $\gamma_1\delta_2 = \mathrm{e}^{\omega_1} g_1 g_2$, but it can be satisfied by a renormalization of polynomials' argument x.

We considered the constraint $\gamma_2 = \delta_1 = 0$ leading to isospectrality. The same picture is reached by the different choice of parameters $\gamma_1 = \delta_2 = 0$, giving the same little q-Jacobi polynomials.

Let us remark that our analysis may be considered as a generalization of the Wynn's work [25], where a general system of orthogonal polynomials has been defined from solutions of the qd-algorithm (isospectral case of the DTTC (1.5), (1.6)). A simple comparison of the above formulae with those of [25] shows that Wynn actually rediscovered the little q-Jacobi polynomials considered by Hahn [11].

5. Analysis of the Askey–Ismail case

In this section we analyse the DTVC solutions for the choice $\zeta(t) \neq \mathrm{e}^{\omega t}$. Because the factor in front of d_n in (3.11) is not zero, we can rewrite this equation as follows

$$d_n = K_1(t)g(x+1) + K_2(t)g(x) + K_3(t)g(x-1) \tag{5.1}$$

where $x = 2n+t$ and the form of $K_i(t)$ is obvious from (3.11). Shifting $t \to t+1$ in (5.1) and subtracting the resulting equation from (5.1) we obtain

$$L_1(t)g(x+2) + L_2(t)g(x+1) + L_3(t)g(x) + L_4(t)g(x-1) = 0 \tag{5.2}$$

where $L_1(t) = K_1(t+1)$, $L_2(t) = K_2(t+1) - K_1(t)$, $L_3(t) = K_3(t+1) - K_2(t)$, $L_4(t) = -K_3(t)$. Equation (5.2) is a simple linear functional equation in two variables x, t. Fixing t, we get an ordinary linear difference equation for $g(x)$ with constant coefficients. Its solution is

$$g(x) = g_1\mathrm{e}^{\omega_1 x} + g_2\mathrm{e}^{\omega_2 x} + g_3\mathrm{e}^{\omega_3 x} \tag{5.3}$$

where g_i and ω_i, $i = 1, 2, 3$ are some constants. Of course, there are some restrictions upon the coefficients $L_i(t)$, however we do not need them here. Substituting (5.3) into (5.1) we find the general expression for d_n

$$d_n = \delta_1\mathrm{e}^{2\omega_1 n} + \delta_2\mathrm{e}^{2\omega_2 n} + \delta_3\mathrm{e}^{2\omega_3 n} \tag{5.4}$$

with some constants δ_i. Substituting the derived forms of d_n and $g(x)$ into (3.5) we find general possible form of $E(t+n)$:

$$E(y) = \epsilon_1\mathrm{e}^{2\omega_1 y} + \epsilon_2\mathrm{e}^{2\omega_2 y} + \epsilon_3\mathrm{e}^{2\omega_3 y}. \tag{5.5}$$

Finally, from (3.5) we arrive at the system of three equations for two unknown functions $\xi(t)$ and $\zeta(t)$:

$$g_i\xi(t) - \delta_i\zeta(t)\mathrm{e}^{-\omega_i(t+1)} = \epsilon_i\mathrm{e}^{\omega_i(t-1)} \qquad i = 1, 2, 3. \tag{5.6}$$

It can be easily shown that this system is compatible only in three cases (defined up to an evident permutation of indices):

(i) $g_3 = 0$, $\omega_3 = (\omega_1 + \omega_2)/2$;

(ii) $\delta_3 = g_3 = \epsilon_3 = 0$;

(iii) $\delta_3 = g_3 = \epsilon_3 = \delta_2 = g_2 = \epsilon_2 = 0$.

Case (i) leads to the condition $\zeta(t) = \zeta_0\mathrm{e}^{2\omega_3 t}$ which is forbidden. Case (iii) is not interesting because it leads to superpotentials A_n, C_n which do not depend on n at all (in this case the polynomials $P_n(x;t)$ can be expressed in terms of the Chebyshev polynomials of the first and second kind). So we only need to consider case (ii). For it, system (5.6) is reduced to two equations with the solution

$$\zeta(t) = \frac{\epsilon_1 g_2 q\mathrm{e}^{(\omega_1+\omega_2)t}(1 - \frac{\epsilon_2 g_1}{\epsilon_1 g_2}q^{t-1})}{\delta_2 g_1(1 - \frac{\delta_1 g_2}{\delta_2 g_1}q^{t+1})} \qquad \xi(t) = \frac{\epsilon_1\mathrm{e}^{\omega_1(t-1)}(1 - \frac{\epsilon_2\delta_1}{\epsilon_1\delta_2}q^{2t})}{g_1(1 - \frac{g_2\delta_1}{g_1\delta_2}q^{t+1})} \tag{5.7}$$

where $q = \exp(\omega_2 - \omega_1)$. From (3.6) we find other unknown functions:

$$c_n = \nu d_{n-1/2} \qquad \eta(t) = \nu\xi(t) \qquad F(y) = \nu E(y + \tfrac{1}{2}) \tag{5.8}$$

where ν is an arbitrary parameter. Taking the special initial condition $A_0(t) = 0$, or $c_0 = 0$, we obtain the constraint $\delta_2 e^{\omega_1} = -\delta_1 e^{\omega_2}$.

The derived $D_{2n}(t)$ and $D_{2n+1}(t)$ allow us to write down recurrence coefficients $v_n(t)$ of the corresponding symmetric OP $S_n(x;t)$ determined from (2.5):

$$v_n(t) = \frac{\nu\delta_1\epsilon_1(1-q^n)(1+\frac{\epsilon_2}{\epsilon_1}q^{n+2t-2})}{e^{2\omega_1}g_1^2(1+\frac{g_2}{g_1}q^{n+t})(1+\frac{g_2}{g_1}q^{n+t-1})}. \tag{5.9}$$

Compare these recurrence coefficients with those determining the Askey–Ismail polynomials analysed in [1, ch 7]:

$$v_n = \frac{(1-q^n)(c-aq^{n-1})}{(c+1-(a+1)q^n)(c+1-(a+1)q^{n-1})} \tag{5.10}$$

where a, c are two independent parameters of the polynomials. It is seen that the constructed coefficients $v_n(t)$ coincide with (5.10) after the identifications

$$\frac{a}{c} = -\frac{\epsilon_2}{\epsilon_1}q^{2t-1} \qquad \frac{a+1}{c+1} = -\frac{g_2}{g_1}q^t. \tag{5.11}$$

The normalization condition $(c+1)^2\nu\delta_1\epsilon_1 = cg_1^2 e^{2\omega_1}$, needed formally for this, can be reached by rescaling the argument x of the polynomials $S_n(x;t)$. For the auxiliary spectral parameter $\kappa(t)$, we then have

$$\kappa^2(t) = \frac{\xi(t)\eta(t)}{\zeta(t)} = -\frac{g_1(1+\frac{\epsilon_2}{\epsilon_1}q^{2t-1})^2}{g_2q^t(1+\frac{g_2}{g_1}q^t)(1-\frac{\epsilon_2 g_1}{\epsilon_1 g_2}q^{t-1})}. \tag{5.12}$$

The set $\kappa(t)$, $t = 0, 1, \ldots$ determines possible points of discrete spectrum of the Askey–Ismail polynomials (see formula (7.51) in [1]). We see that there are just three relevant parameters ϵ_1/ϵ_2, g_1/g_2 and q, the others being pure normalization factors.

We may conclude that the choice $\zeta(t) \neq e^{\omega t}$ corresponds to general symmetric Askey–Ismail polynomials expressed through a ${}_3\varphi_2$ basic series [1]. Note that one can also construct the non-symmetric orthogonal polynomials $P_n(x;t)$ with the help of relations (2.11), which will be different from the q-Pollaczek polynomials [4]. The theorem is proved completely, i.e. recurrence coefficients of the most general classical OP and of another very general class of explicitly known OP arise very naturally as similarity solutions of the DTVC.

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